Semigroup Forum (2012) 85:513–524
DOI 10.1007/s00233-012-9430-2

RESEARCH ARTICLE

# Automorphisms of partition order-decreasing transformation monoids

**Haobo Yang · Xiuliang Yang**

Received: 24 January 2010 / Accepted: 18 August 2012 / Published online: 13 September 2012


**Abstract** Let $T_n$ be the full transformation semigroup on a finite set $X_n = \{1, 2, \dots, n\}$. Let $\rho$ be an equivalence relation on $X_n$ and $\preceq$ be a total order on the partition set $X_n/\rho$. We describe all automorphisms of the partition order-decreasing transformation monoid:

$$T(\rho, \preceq) = \left\{\alpha \in T_n : (x\alpha)\rho \preceq x\rho, \forall x \in X_n\right\}$$

that generalizes the results of Schreier (Fundam. Math., 28:261–264, 1936) and Šutov (Izv. Vysš. Učebn. Zaved., Mat., 3:177–184, 1961).

**Keywords** Transformation semigroup · Partition order-decreasing · Automorphism group

## 1 Introduction

For the standard definitions on semigroups and transformation semigroups we refer the reader to the books [3–5].

Let $X_n = \{1, 2, \dots, n\}$. A submonoid $M$ of the semigroup $T_n$ of full transformations on $X_n$ is *intransitive* if there exist $x, y$ in $X_n$ such that $(x)\varphi \neq y$ for any $\varphi \in M$.

---

Communicated by Jean-Eric Pin.

This work is supported by NSF (China) grant no. 10971086.

H. Yang
Qiangjiang College, Hangzhou Normal University, Hangzhou City, Zhejiang 310012, P.R. China
e-mail: yanghb@hznu.edu.cn

X. Yang (✉)
Department of Mathematics, Hangzhou Normal University, Xiasha, Hangzhou City, Zhejiang 310036, P.R. China
e-mail: yxl@hznu.edu.cn

A submonoid $M$ of $T_n$ said to be *half-transitive* provided that it is intransitive, and for every ordered pair $(x, y) \in X_n \times X_n$ there is some $\varphi \in M$ such that either $x\varphi = y$ or $y\varphi = x$. In [10] we showed that, for every half-transitive submonoid $M$ of $T_n$, there exist a non-universal equivalence relation $\rho$ on $X_n$ and a total order $\preceq$ on the partition set $X_n/\rho$ such that $M$ lies inside a half-transitive submonoid $T(\rho, \preceq)$ of $T_n$ defined by

$$T(\rho, \preceq) = \big\{\alpha \in T_n : (x\alpha)\rho \preceq x\rho, \forall x \in X_n\big\}.$$

Here we consider the monoid $T(\rho, \preceq)$ for arbitrary equivalence relation $\rho$ on $X_n$. In particular, if $\rho$ is universal then $T(\rho, \preceq) = T_n$; $T(\rho, \preceq)$ is the order-decreasing finite full transformation monoid if $\rho$ is the identity relation [9]. If $X_n/\rho = \{\{1\}, X_n - \{1\}\}$ and $\{1\} \prec X_n - \{1\}$ then $T(\rho, \preceq)$ is isomorphic to $PT_{n-1}$, the semigroup of partial transformations of $X_n - \{1\}$.

Note that automorphisms of $T_n$ and $PT_{n-1}$ were described by Schreier and Šutov in [6] and [8] respectively. In present paper, we will describe automorphisms of the monoid $T(\rho, \preceq)$. Thus our result generalizes the results of Schreier [6] and Šutov [8].

For the topics of automorphisms of transformation semigroups we refer the reader to the references of [1] and [3]. We point out that $T(\rho, \preceq)$ is not the centralizer of any idempotent in $T_n$ [1] and, in general, $T(\rho, \preceq)$ does not contain all constant transformations of $X_n$. Our method is different from any one of the references of [1] and [7]. Here we use a basis fact that an automorphism $\varphi$ of a monoid $S$ maps units of $S$ to its units, and idempotents of $S$ to its idempotents.

To achieve our aims, the organization of the paper is as follows: In Sect. 2 we investigate the units and the idempotents of $T(\rho, \preceq)$, and describe the generators of $T(\rho, \preceq)$. In Sect. 3 we describe the Green's $^*$-relations $\mathcal{L}^*$ and $\mathcal{R}^*$ on $T(\rho, \preceq)$. As a consequence, $T(\rho, \preceq)$ is shown to be abundant. Finally, in Sect. 4, we use the results of the previous sections to determine all automorphisms of the monoid $T(\rho, \preceq)$.

Throughout the paper, we use the following notations: let $|A|$ denote the cardinality of a set $A$. $1_A$ denote the identity function from $A$ to itself. For a function $\alpha : A \to B$, denote the image of $\alpha$ by $\operatorname{im}\alpha$. $|\operatorname{im}\alpha|$ is said to be the *rank* of $\alpha$, and we can write

$$\alpha = \begin{pmatrix} A_1 & \dots & A_k \\ a_1 & \dots & a_k \end{pmatrix},$$

where $\operatorname{im}\alpha = \{a_1, \dots, a_k\}$, $a_i\alpha^{-1} = A_i$ $(i = 1, \dots, k)$ and $\{A_1, \dots, A_k\} = A/\ker\alpha$, where $\ker\alpha$ is the kernel of $\alpha$ (the equivalence relation $\{(x, y) \in A \times A : x\alpha = y\alpha\}$). Denote the restriction of $\alpha$ to $C$ by $\alpha|_C$ for $C \subseteq A$ and $(C)\alpha$ denotes the image set of $C$ under $\alpha$.

Let $E_{n-1}$ denote the set of idempotents in $T_n$ of rank $n - 1$. Every element $e$ of $E_{n-1}$ has the form

$$e = \begin{pmatrix} b \\ a \end{pmatrix}$$

for some $a, b \in X_n, a \neq b$, which maps $b$ to $a$ and $x$ to itself for any $x \in X_n - \{b\}$ [4].

For the remainder of the paper, $\rho$ will denote an equivalence relation on $X_n$ and $\preceq$ will denote a total order on the partition set $X_n/\rho$.

## 2 Units, idempotents and generators

In this section we describe the units and the idempotents of $T(\rho, \preceq)$, and determine the generators of $T(\rho, \preceq)$.

We define

$$U_\rho = \{\alpha \in T_n : (x\rho)\alpha = x\rho, x \in X_n\}.$$

Clearly, $U_\rho \subseteq T(\rho, \preceq)$. Let $S_n$ be the symmetric group on $X_n$. The following lemma gives the description of the group of units of $T(\rho, \preceq)$.

**Lemma 2.1** *Let $\alpha \in T_n$. Then the following statements are equivalent*:

(1) *$\alpha$ is a unit of $T(\rho, \preceq)$.*
(2) *$\alpha \in S_n \cap T(\rho, \preceq)$.*
(3) *$\alpha \in U_\rho$.*

*Proof* Obviously (1) $\Rightarrow$ (2).

(2) $\Rightarrow$ (3). Suppose that $\alpha \in S_n \cap T(\rho, \preceq)$. Then $\alpha^{-1} \in S_n \cap T(\rho, \preceq)$. Given any $x \in X_n$. Take any $y \in x\rho$, we have

$$y\rho = (y\alpha\alpha^{-1})\rho \preceq (y\alpha)\rho \preceq y\rho = x\rho,$$

and so $y\alpha \in x\rho$. Thus $(x\rho)\alpha \subseteq x\rho$. To prove that $x\rho \subseteq (x\rho)\alpha$ consider $z \in x\rho$. Then we have

$$x\rho = z\rho = [(z\alpha^{-1})\alpha]\rho \preceq (z\alpha^{-1})\rho \preceq z\rho$$

and so $z\alpha^{-1} \in x\rho$ and $z = (z\alpha^{-1})\alpha$. It follows that $(x\rho)\alpha = x\rho$ and hence $\alpha \in U_\rho$.

(3) $\Rightarrow$ (1). By the definition of $U_\rho$, we have that $\alpha \in T(\rho, \preceq)$ and the restriction $\alpha|_{x\rho}$ is a bijection from the $\rho$-class $x\rho$ onto itself, and so $\alpha \in S_n$ and $\alpha^{-1} \in T(\rho, \preceq)$. Thus $\alpha$ is a unit of $T(\rho, \preceq)$. □

Next we have

**Lemma 2.2** *Let $\alpha \in T(\rho, \preceq)$. Then $\alpha$ is an idempotent if and only if for all $t \in \operatorname{im}\alpha$, $t\alpha = t$ and $t\rho = \min\{x\rho : x \in t\alpha^{-1}\}$.*

*Proof* First recall that $\alpha \in T_n$ is an idempotent if and only if, for all $t \in \operatorname{im}\alpha$, $t \in t\alpha^{-1}$. Hence $\alpha \in T(\rho, \preceq)$ is an idempotent if and only if, for all $t \in \operatorname{im}\alpha$,

$$t\alpha = t \quad \text{and} \quad t\rho = \min\{x\rho : x \in t\alpha^{-1}\}$$

since $t \in t\alpha^{-1}$ and $x \in t\alpha^{-1}$ implies that $t = t\alpha = x\alpha$ and $t\rho = (x\alpha)\rho \preceq x\rho$. □

**Corollary 2.3** *Let $a, b \in X_n$ and $a \neq b$. Then $a\rho \preceq b\rho$ if and only if $\binom{b}{a} \in T(\rho, \preceq)$.*

From Corollary 2.3 we introduce the following idempotent subset of $T(\rho, \preceq)$ with rank $n-1$ (that is often used in the paper):

$$E_\rho^w = \left\{ \begin{pmatrix} b \\ a \end{pmatrix} \in E_{n-1} : a\rho \preceq b\rho \right\}.$$

Then we have

**Lemma 2.4** *Let $\alpha$ be an element of $T(\rho, \preceq) - U_\rho$. Then $\alpha$ is a product of elements in $E_\rho^w \cup U_\rho$.*

*Proof* Let $s(\alpha)$ be the cardinality of the set $\{x \in X_n : x\alpha \neq x\}$. We will show that $\alpha$ can be written as a product of elements in $E_\rho^w \cup U_\rho$ by using induction on $s(\alpha)$.

Clearly, if $s(\alpha) = 1$ then $\alpha \in E_\rho^w$. We now assume that $s(\alpha) > 1$ and let

$$\alpha = \begin{pmatrix} A_1 & A_2 & \dots & A_k \\ a_1 & a_2 & \dots & a_k \end{pmatrix} \in T(\rho, \preceq) - U_\rho.$$

Then, for every $i$, $a_i\rho \preceq \min\{x\rho : x \in A_i\}$ from the definition of $T(\rho, \preceq)$. Choose $b_i \in A_i$ such that $b_i\rho = \min\{x\rho : x \in A_i\}$ for every $i$. Without loss of generality we may assume that

$$a_1 = b_1, \dots, a_i = b_i, \qquad a_{i+1} \neq b_{i+1}, \dots, a_k \neq b_k.$$

Define

$$\beta = \begin{pmatrix} A_1 & \dots & A_k \\ b_1 & \dots & b_k \end{pmatrix} \quad \text{and} \quad \gamma = \begin{pmatrix} b_1 & \dots & b_i & b_{i+1} & \dots & b_k & Y \\ b_1 & \dots & b_i & a_{i+1} & \dots & a_k & y \end{pmatrix},$$

where $Y = X_n - \{b_1, \dots, b_k\}$, $y \in Y$ and $y\rho = \min\{z\rho : z \in Y\}$. Then $\beta \in T(\rho, \preceq) - U_\rho$ and $\gamma \in T(\rho, \preceq)$. Moreover, $\gamma \in U_\rho$ if and only if $k = n-1$.

Note that $s(\alpha) = n - i > n - k = s(\beta)$ and $s(\gamma) = s(\alpha) - 1$. Thus both $\beta$ and $\gamma$ are products of elements in $E_\rho^w \cup U_\rho$ by inductive supposition. It follows that $\alpha = \beta\gamma$ is a product of elements in $E_\rho^w \cup U_\rho$. This completes the proof. □

From Lemma 2.4 we immediately deduce

**Proposition 2.5** *Let $U_\rho$ and $E_\rho^w$ be defined above. Then $E_\rho^w \cup U_\rho$ is a generating set of $T(\rho, \preceq)$.*

Similar to the proof as Lemma 2.4 we also have

**Lemma 2.6** *Let $\alpha$ be an idempotent of $T(\rho, \preceq) - U_\rho$. Then $\alpha$ is a product of elements in $E_\rho^w$.*

## 3 Green's *-relations $\mathcal{L}^*$ and $\mathcal{R}^*$

We recall some of the basis facts about Green's *-relations $\mathcal{L}^*$ and $\mathcal{R}^*$. The relations $\mathcal{L}^*(\mathcal{R}^*)$ is defined on a semigroup $S$ by the rule $a\mathcal{L}^*b(a\mathcal{R}^*b)$ if and only if the elements $a, b$ of $S$ are related by the Green's relation $\mathcal{L}(\mathcal{R})$ in some oversemigroup of $S$ [2]. The following lemma, from [2], provides us an alternative description for $\mathcal{L}^*(\mathcal{R}^*)$.

**Lemma 3.1** *Let $S$ be a semigroup and let $a, b$ be in $S$. The following conditions are equivalent*:

(1) $a\mathcal{L}^*b(a\mathcal{R}^*b)$.
(2) *for all* $s, t \in S^1$, $as = at\,(sa = ta)$ *if and only if* $bs = bt\,(sb = tb)$.

For $\alpha \in T_n$, $\ker\alpha = \{(x, y) \in X_n \times X_n : x\alpha = y\alpha\}$. Then we have

**Proposition 3.2** *Let $\alpha, \beta$ be elements of $T(\rho, \preceq)$. Then*

(1) $(\alpha, \beta) \in \mathcal{L}^*$ *if and only if* $\operatorname{im}\alpha = \operatorname{im}\beta$;
(2) $(\alpha, \beta) \in \mathcal{R}^*$ *if and only if* $\ker\alpha = \ker\beta$.

*Proof* (1) Certainly if $\operatorname{im}\alpha = \operatorname{im}\beta$ then $(\alpha, \beta) \in \mathcal{L}(T_n)$ [5] and so $(\alpha, \beta) \in \mathcal{L}^*$.
Conversely, suppose that $(\alpha, \beta) \in \mathcal{L}^*$. Then

$$\alpha\gamma = \alpha\delta \quad \text{if and only if} \quad \beta\gamma = \beta\delta \quad \big(\text{for all } \gamma, \delta \in T(\rho, \preceq)\big).$$

Let $X_n/\rho = \{Y_1 \prec Y_2 \prec \cdots \prec Y_t\}$. If $Y_1 = \{x\}$, for some $x \in X_n$, then, certainly $x\alpha = x = x\beta$; Otherwise, choose $y \in Y_1$ and $y \neq x$, then $\binom{x}{y} \in T(\rho, \preceq)$. Hence

$$x \notin \operatorname{im}\alpha \Leftrightarrow \alpha \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \alpha \cdot 1_{X_n} \Leftrightarrow \beta \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \beta \cdot 1_{X_n} \Leftrightarrow x \notin \operatorname{im}\beta.$$

We therefore conclude that $\operatorname{im}\alpha = \operatorname{im}\beta$.
(2) Again if $\ker\alpha = \ker\beta$ then $(\alpha, \beta) \in \mathcal{R}(T_n)$ [5] and so $(\alpha, \beta) \in \mathcal{R}^*$.
Conversely, if $(\alpha, \beta) \in \mathcal{R}^*$ then

$$\gamma\alpha = \delta\alpha \Leftrightarrow \gamma\beta = \delta\beta \quad \big(\text{for all } \gamma, \delta \in T(\rho, \preceq)\big).$$

For $x, y \in X_n$ with $x \neq y$, we may assume that $y\rho \preceq x\rho$. Then $\binom{x}{y} \in T(\rho, \preceq)$. Hence

$$x\alpha = y\alpha \Leftrightarrow \begin{pmatrix} x \\ y \end{pmatrix} \cdot \alpha = 1_{X_n} \cdot \alpha \Leftrightarrow \begin{pmatrix} x \\ y \end{pmatrix} \cdot \beta = 1_{X_n} \cdot \beta \Leftrightarrow x\beta = y\beta.$$

We therefore conclude that $\ker\alpha = \ker\beta$. □

A semigroup $S$ in which each $\mathcal{L}^*$-class and each $\mathcal{R}^*$-class contains an idempotent is called *abundant* [2]. We have

**Corollary 3.3** *For any equivalence relation $\rho$ on $X_n$ and a total order $\preceq$ on the partition set $X_n/\rho$, the semigroup $T(\rho, \preceq)$ is abundant.*

*Proof* For a typical $\mathcal{R}^*$-class $R^*_\alpha$ of $T(\rho, \preceq)$ with $X_n/\ker\alpha = \{A_1, A_2, \ldots, A_k\}$. Choose $b_i \in A_i$ such that $b_i\rho = \min\{x_i\rho : x_i \in A_i\}$ for every $i \in \{1, \ldots, k\}$. Then we have that

$$\beta = \begin{pmatrix} A_1 & \ldots & A_k \\ b_1 & \ldots & b_k \end{pmatrix}$$

is an idempotent of $T(\rho, \preceq)$ by Lemma 2.2. and $\alpha R^*\beta$ by Proposition 3.2.

Next, consider that a typical $\mathcal{L}^*$-class $L^*_\alpha$ of $T(\rho, \preceq)$ with $\operatorname{im}\alpha = \{a_1, \ldots, a_k\}$, where $1 \leq k \leq n$. We will use induction on $k$ to show that $L^*_\alpha$ contains an idempotent of $T(\rho, \preceq)$. If $k = 1$ then, clearly $\alpha$ is an idempotent. Suppose now that the conclusion holds for $k - 1$. Consider that

$$\alpha = \begin{pmatrix} A_1 & \ldots & A_{k-1} & A_k \\ a_1 & \ldots & a_{k-1} & a_k \end{pmatrix} \in T(\rho, \preceq).$$

Without loss of generality, we may assume that

$$a_1\rho \preceq \cdots \preceq a_{k-1}\rho \preceq a_k\rho.$$

Then $\binom{a_k}{a_{k-1}} \in T(\rho, \preceq)$ by Corollary 2.3, and so

$$\beta = \begin{pmatrix} A_1 & \ldots & A_{k-2} & A_{k-1} \cup A_k \\ a_1 & \ldots & a_{k-2} & a_{k-1} \end{pmatrix} = \alpha \begin{pmatrix} a_k \\ a_{k-1} \end{pmatrix} \in T(\rho, \preceq).$$

By induction supposition, there exists an idempotent $\varepsilon \in T(\rho, \preceq)$ such that $\beta\mathcal{L}^*\varepsilon$. Hence, by Lemma 2.2, we can let

$$\varepsilon = \begin{pmatrix} B_1 & \ldots & B_{k-2} & B_{k-1} \\ a_1 & \ldots & a_{k-2} & a_{k-1} \end{pmatrix}$$

with $a_i \in B_i, i = 1, \ldots, k-1$ and $a_i\rho = \min\{x_i\rho : x_i \in B_i\}$ for every $i \in \{1 \ldots, k-1\}$. Since $a_k \in X_n = B_1 \cup \cdots \cup B_{k-1}$, there exist a unique $B_j$ such that $a_k \in B_j$. Note that $a_k \neq a_j$, so $a_j \in B_j \backslash \{a_k\}$. It follows, by Lemma 2.2, that

$$\varepsilon^* = \begin{pmatrix} B_1 & \ldots & B_{j-1} & B_j\backslash\{a_k\} & B_{j+1} & \ldots & B_{k-1} & a_k \\ a_1 & \ldots & a_{j-1} & a_j & a_{j+1} & \ldots & a_{k-1} & a_k \end{pmatrix}$$

is an idempotent in $T(\rho, \preceq)$, so that $\alpha\mathcal{L}^*\varepsilon^*$ as required. □

## 4 Automorphisms of $T(\rho, \preceq)$

After the preliminaries of the previous two sections, to determine every automorphism of $T(\rho, \preceq)$, we need the following lemmas.

**Lemma 4.1** *Let $\binom{b_1}{a_1}, \binom{b_2}{a_2} \in E_{n-1}$. Then $\binom{b_1}{a_1}\binom{b_2}{a_2} = \binom{b_1}{a_1}$ if and only if $b_1 = b_2$.*

*Proof* Assume that $b_1 = b_2$. Then, from $a_1 \neq b_1$ we have

$$b_1\binom{b_1}{a_1}\binom{b_1}{a_2} = a_1\binom{b_1}{a_2} = a_1 = b_1\binom{b_1}{a_1},$$

and for every $x \in X_n\backslash\{b_1\}$,

$$x\binom{b_1}{a_1}\binom{b_1}{a_2} = x\binom{b_1}{a_2} = x = x\binom{b_1}{a_1}$$

and so $\binom{b_1}{a_1}\binom{b_1}{a_2} = \binom{b_1}{a_1}$.

Assume now that $b_1 \neq b_2$. Then

$$b_2\binom{b_1}{a_1}\binom{b_2}{a_2} = b_2\binom{b_2}{a_2} = a_2 \neq b_2 = b_2\binom{b_1}{a_1},$$

and so $\binom{b_1}{a_1}\binom{b_2}{a_2} \neq \binom{b_1}{a_1}$. □

Similar argument as in Lemma 4.1, we have

**Lemma 4.2** *Let* $\binom{x}{y}, \binom{y}{w} \in E_{n-1}$. *Then* $\binom{y}{w}\binom{x}{y}$ *is an idempotent if and only if* $x = w$.

**Lemma 4.3** *Let* $\binom{b_1}{a_1}, \binom{b_2}{a_2} \in T(\rho, \preceq)$ *and* $b_1\rho \preceq b_2\rho$. *We have*

(1) *If* $b_1 = b_2$ *then* $\binom{b_1}{a_1}\mathcal{L}^*\binom{b_2}{a_2}$.
(2) *If* $b_1 \neq b_2$ *then there exists* $\delta \in T(\rho, \preceq)$ *such that* $\binom{b_1}{a_1}\mathcal{R}^*\delta\mathcal{L}^*\binom{b_2}{a_2}$.

*Proof* Since $\operatorname{im}\binom{b_1}{a_1} = X_n\backslash\{b_1\} = \operatorname{im}\binom{b_1}{a_2}$, so (1) holds by Lemma 3.1.

For (2), note that $\binom{b_2}{b_1} \in T(\rho, \preceq)$ by Corollary 2.3, since $b_1\rho \preceq b_2\rho$. It follows that $\delta = \binom{b_1}{a_1}\binom{b_2}{b_1} \in T(\rho, \preceq)$ with $\ker\delta = \ker\binom{b_1}{a_1}$ and $\operatorname{im}\delta = \operatorname{im}\binom{b_2}{b_1} = \operatorname{im}\binom{b_2}{a_2}$. Hence, by Proposition 3.2, we have $\binom{b_1}{a_1}\mathcal{R}^*\delta\mathcal{L}^*\binom{b_2}{a_2}$. □

Let $\operatorname{Aut} T(\rho, \preceq)$ be the automorphism group of $T(\rho, \preceq)$. Then we have

**Lemma 4.4** *Let* $\varphi \in \operatorname{Aut} T(\rho, \preceq)$. *Then* $(E_\rho^w)\varphi = E_\rho^w$.

*Proof* First, we claim that

$$(E_\rho^w)\varphi \cap E_\rho^w \neq \emptyset.$$

Assume that it is not such case. Note that $(\varepsilon_i)\varphi$ is idempotent for every $\varepsilon_i \in E_\rho^w$, and so we certainly have $|\operatorname{im}(\varepsilon_i)\varphi| \leq n-2$. Next, given an element $\varepsilon \in E_\rho^w$ then there exists an idempotent $\omega \in T(\rho, \preceq) - \{1_{X_n}\}$ such that $(\omega)\varphi = \varepsilon$. Thus, by Lemma 2.6, there exist elements $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k$ of $E_\rho^w$ such that $\omega = \varepsilon_1\varepsilon_2\cdots\varepsilon_k$. Hence we have

$$\varepsilon = (\varepsilon_1\varepsilon_2\cdots\varepsilon_k)\varphi = (\varepsilon_1)\varphi(\varepsilon_2)\varphi\cdots(\varepsilon_k)\varphi.$$

It follows that $n-1 = |\operatorname{im}\varepsilon| \leq |\operatorname{im}(\varepsilon_k)\varphi| \leq n-2$ (since $(\varepsilon_k)\varphi \notin E_\rho^w$), a contradiction.

We have shown that $(E_\rho^w)\varphi \cap E_\rho^w \neq \emptyset$. Next, let $\varepsilon_0 = \binom{b_0}{a_0} = (\binom{b_1}{a_1})\varphi \in (E_\rho^w)\varphi \cap E_\rho^w$. For any $\varepsilon = \binom{b}{a} \in E_\rho^w$, we distinguish two cases as follows.

Case 1: $b = b_1$. In this case, by (1) of Lemma 4.3 we have $\varepsilon \mathcal{L}^* \binom{b_1}{a_1}$. Further, by Lemma 3.1, we immediately deduce $(\varepsilon)\varphi \mathcal{L}^* \varepsilon_0$. Hence, by (1) of Proposition 3.2, $|\operatorname{im}(\varepsilon)\varphi| = n-1$ and so $(\varepsilon)\varphi \in E_\rho^w$.

Case 2: $b \neq b_1$; $b\rho \preceq b_1\rho$(Similar argument for $b_1\rho \preceq b\rho$). By (2) of Lemma 4.3, there exists $\delta \in T(\rho, \preceq)$ such that $\varepsilon \mathcal{R}^* \delta \mathcal{L}^* \binom{b_1}{a_1}$. Hence, by Lemma 3.1 we have $(\varepsilon)\varphi \mathcal{R}^* (\delta)\varphi \mathcal{L}^* \varepsilon_0$. That is, $\ker(\varepsilon)\varphi = \ker(\delta)\varphi$ and $\operatorname{im}(\delta)\varphi = \operatorname{im}\varepsilon_0$. It follows that

$$\left|\operatorname{im}(\varepsilon)\varphi\right| = \left|X_n/\ker(\varepsilon)\varphi\right| = \left|X_n/\ker(\delta)\varphi\right| = \left|\operatorname{im}(\delta)\varphi\right| = |\operatorname{im}\varepsilon_0| = n-1,$$

and so $(\varepsilon)\varphi \in E_\rho^w$.

By Case 1 and Case 2 above, we have shown that $(E_\rho^w)\varphi \subseteq E_\rho^w$. By using the foregoing argument for the automorphism $\varphi^{-1}$, we have $(E_\rho^w)\varphi^{-1} \subseteq E_\rho^w$. It follows that

$$E_\rho^w = \left(E_\rho^w\right)\varphi^{-1}\varphi \subseteq \left(E_\rho^w\right)\varphi \subseteq E_\rho^w,$$

and so $(E_\rho^w)\varphi = E_\rho^w$. □

**Lemma 4.5** *Let $\varphi \in \operatorname{Aut} T(\rho, \preceq)$. Then there exists $\mu_\varphi \in U_\rho$ such that $(\binom{x}{y})\varphi = \binom{x\mu_\varphi}{y\mu_\varphi}$ for every $\binom{x}{y} \in E_\rho^w$.*

*Proof* Let $X_n/\rho = \{Y_1 \prec Y_2 \prec \cdots \prec Y_t\}$. We distinguish two cases: $|Y_1| = 1$ or $|Y_1| \geq 2$.

Case 1: $|Y_1| = 1$. Let $Y_1 = \{1\}$. For any $x \in X_n \backslash \{1\}$, clearly $\binom{x}{1} \in E_\rho^w$ and by Lemma 4.4 we can let $(\binom{x}{1})\varphi = \binom{x'}{1'_x} \in E_\rho^w$. We will prove that $1 = 1'_x$. Indeed, if $1 \neq 1'_x$ then $\binom{1'_x}{1} \in E_\rho^w$, and by Lemma 4.4 we can let $(\binom{z}{t})\varphi = \binom{1'_x}{1}$, where $\binom{z}{t} \in E_\rho^w$. Since $z \neq t$ and $Y_1 \preceq t\rho \preceq z\rho$, we have $z \neq 1$ and so

$$\binom{z}{t}\binom{x}{1} = \begin{cases} \binom{x}{1}, & t = 1, x = z \\ \binom{z}{t}, & t \neq 1, x = z \\ \begin{pmatrix} \{z, x, 1\} & k & \ldots \\ 1 & k & \ldots \end{pmatrix}, & t = 1, x \neq z \\ \begin{pmatrix} \{z, t\} & \{x, 1\} & k & \ldots \\ t & 1 & k & \ldots \end{pmatrix}, & t \neq 1, x \neq z \end{cases}$$

is an idempotent, but $\binom{1'_x}{1}\binom{x'}{1'_x}$ is not idempotent by Lemma 4.2, a contradiction. Thus $1 = 1'_x$ and we have proved that

$$\left(\binom{x}{1}\right)\varphi = \binom{x'}{1} \quad \text{for any } x \in X_n \backslash \{1\}.$$

Now we define a mapping $\mu_\varphi : X_n \to X_n$ by $1\mu_\varphi = 1$ and $x\mu_\varphi = x'$ (defined the above), $x \neq 1$. Obviously, $\mu_\varphi$ is a bijection.

Next, we will prove that $\left(\binom{x}{y}\right)\varphi = \binom{x\mu_\varphi}{y\mu_\varphi}$ for any $\binom{x}{y} \in E^w_\rho, x \neq 1, y \neq 1$. By Lemma 4.4 we can suppose that $\left(\binom{x}{y}\right)\varphi = \binom{x^*}{y^*} \in E^w_\rho$. Since

$$\binom{x\mu_\varphi}{1} = \left(\binom{x}{1}\right)\varphi = \left(\binom{x}{1}\binom{x}{y}\right)\varphi = \binom{x\mu_\varphi}{1}\binom{x^*}{y^*},$$

we immediately deduce that $x^* = x\mu_\varphi$ by Lemma 4.1. Moreover,from $\left(\binom{y}{1}\binom{x}{y}\right)\varphi = \binom{y\mu_\varphi}{1}\binom{x\mu_\varphi}{y^*}$ we see that if $y\mu_\varphi \neq y^*$ then $\binom{y\mu_\varphi}{1}\binom{x\mu_\varphi}{y^*}$ is an idempotent, and so $\binom{y}{1}\binom{x}{y}$ is idempotent (since $\varphi$ is an automorphism). But this is impossible since $\binom{y}{1}\binom{x}{y}$ is not an idempotent. Hence $y\mu_\varphi = y^*$. It follows that if $y\rho \preceq x\rho$ then $(y\mu_\varphi)\rho \preceq (x\mu_\varphi)\rho$, and so $\mu_\varphi \in U_\rho$.

Case 2: $|Y_1| \geq 2$. In this case, for any $x \in X_n$, there exists $y \in X_n \backslash \{x\}$ such that $y\rho \preceq x\rho$, and so $\binom{x}{y} \in E^w_\rho$. By Lemma 4.4 we can let

$$\left(\binom{x}{y}\right)\varphi = \binom{x'}{y'} \in E^w_\rho \quad \text{and} \quad \left(\binom{x}{z}\right)\varphi = \binom{x'_z}{z'} \in E^w_\rho.$$

Note that

$$\binom{x'}{y'} = \left(\binom{x}{y}\right)\varphi = \left[\left(\binom{x}{y}\binom{x}{z}\right)\right]\varphi = \binom{x'}{y'}\binom{x'_z}{z'},$$

and so $x' = x'_z$ by Lemma 4.1. Hence, $\varphi$ induces a map $\mu_\varphi$ from $X_n$ to itself, defined by

$$\left(\binom{x}{y}\right)\varphi = \binom{x\mu_\varphi}{y'}, \quad \text{for every } \binom{x}{y} \in E^w_\rho.$$

Obviously, $\mu_\varphi$ is surjective.

**Fact 1** $\mu_\varphi$ is injective.

Indeed, assume that $t = x_1\mu_\varphi = x_2\mu_\varphi$, for $x_1, x_2 \in X_n$. Since $|Y_1| \geq 2$, there exist $y_1, y_2 \in X_n$ such that $\binom{x_1}{y_1}, \binom{x_2}{y_2} \in E^w_\rho$. By Lemma 4.4 we let $\left(\binom{x_1}{y_1}\right)\varphi = \binom{t}{y'_1}$ and $\left(\binom{x_2}{y_2}\right)\varphi = \binom{t}{y'_2}$. Then

$$\left(\binom{x_1}{y_1}\binom{x_2}{y_2}\right)\varphi = \binom{t}{y'_1}\binom{t}{y'_2} = \binom{t}{y'_1} = \left(\binom{x_1}{y_1}\right)\varphi.$$

As $\varphi$ is a bijection it follows that $\binom{x_1}{y_1}\binom{x_2}{y_2} = \binom{x_1}{y_1}$, and so $x_1 = x_2$ from Lemma 4.1.

**Fact 2** If $x_1\rho \preceq x_2\rho$. Then $(x_1\mu_\varphi)\rho \preceq (x_2\mu_\varphi)\rho$.

In fact, assume that $(x_1\mu_\varphi)\rho \succ (x_2\mu_\varphi)\rho$. Then $\binom{x_2}{x_1}, \binom{x_1\mu_\varphi}{x_2\mu_\varphi} \in E^w_\rho$. By Lemma 4.4 we can let

$$\left(\binom{x_2}{x_1}\right)\varphi = \binom{x_2\mu_\varphi}{x_1'}, \qquad \left(\binom{x_1}{z}\right)\varphi = \binom{x_1\mu_\varphi}{x_2\mu_\varphi}.$$

Note that $\binom{x_1\mu_\varphi}{x_2\mu_\varphi}\binom{x_2\mu_\varphi}{x_1'}$ is an idempotent. Hence $\binom{x_1}{z}\binom{x_2}{x_1}$ is also an idempotent, and so $z = x_2$ by Lemma 4.2. Further,

$$\binom{x_1\mu_\varphi}{x_2\mu_\varphi} = \left(\binom{x_1}{x_2}\right)\varphi = \left(\binom{x_2}{x_1}\binom{x_1}{x_2}\right)\varphi = \binom{x_2\mu_\varphi}{x_1'}\binom{x_1\mu_\varphi}{x_2\mu_\varphi}$$

is an idempotent. Hence, $x_1' = x_1\mu_\varphi$ by Lemma 4.2. Since now both $\binom{x_2\mu_\varphi}{x_1\mu_\varphi}$ and $\binom{x_1\mu_\varphi}{x_2\mu_\varphi}$ belong to $E^w_\rho$, we have $(x_1\mu_\varphi)\rho = (x_2\mu_\varphi)\rho$, which contradicts the assumption.

**Fact 3** Let $(\binom{x}{y})\varphi = \binom{x\mu_\varphi}{y'}$ for $\binom{x}{y} \in E^w_\rho$. Then $y' = y\mu_\varphi$.

Since $\binom{x}{y} \in E^w_\rho$,we have $y\rho \preceq x\rho$, and so $\binom{x\mu_\varphi}{y\mu_\varphi} \in E^w_\rho$ by Facts 2 and 1. By Lemma 4.4 we can let

$$\left(\binom{x}{z}\right)\varphi = \binom{x\mu_\varphi}{y\mu_\varphi} \quad \text{for some} z \in X_n\backslash\{x\}.$$

It is sufficient to show that $y = z$. Assume now that $y \neq z$. Then $y\rho \preceq z\rho$ or $z\rho \preceq y\rho$ since $\preceq$ is a totally order.

For the former, by Facts 2 and 1 we have $\binom{z\mu_\varphi}{y\mu_\varphi} \in E^w_\rho$ and by Lemma 4.4 we can let $\binom{z\mu_\varphi}{y\mu_\varphi} = (\binom{z}{t})\varphi$, for some $t \in X_n$. Note that $\binom{z\mu_\varphi}{y\mu_\varphi}\binom{x\mu_\varphi}{y\mu_\varphi}$ is idempotent. Then $\binom{z}{t}\binom{x}{z}$ is also idempotent, and so $t = x$ by Lemma 4.2. Hence

$$\binom{z\mu_\varphi}{y\mu_\varphi} = \left(\binom{z}{x}\right)\varphi = \left(\binom{x}{z}\binom{z}{x}\right)\varphi = \binom{x\mu_\varphi}{y\mu_\varphi}\binom{z\mu_\varphi}{y\mu_\varphi}.$$

It follows that $x\mu_\varphi = z\mu_\varphi$ and so $x = z$, which contradicts $x \neq z$.

For the latter,then $\binom{y}{z} \in E^w_\rho$ and by Lemma 4.4 we can let $(\binom{y}{z})\varphi = \binom{y\mu_\varphi}{z'}$ for some $z' \in X_n$. Note that $\binom{y}{z}\binom{x}{z}$ is idempotent. We have that $\binom{y\mu_\varphi}{z'}\binom{x\mu_\varphi}{y\mu_\varphi}$ is also idempotent, and so $z' = x\mu_\varphi$ by Lemma 4.2. Hence, from

$$\left(\binom{x}{z}\binom{y}{z}\right)\varphi = \binom{x\mu_\varphi}{y\mu_\varphi}\binom{y\mu_\varphi}{x\mu_\varphi} = \binom{y\mu_\varphi}{x\mu_\varphi} = \left(\binom{y}{z}\right)\varphi$$

and $\varphi$ is a bijection, we have that $\binom{x}{z}\binom{y}{z} = \binom{y}{z}$ and so $x = y$, a contradiction.

We have proved Fact 3, and from Fact 2 we see that $\mu_\varphi \in U_\rho$, and so the proof of Lemma 4.5 is now completes. □

**Lemma 4.6** *For any $\binom{x}{y} \in E^w_\rho$ and for any $\mu \in U_\rho$ we have that*

$$\mu^{-1}\binom{x}{y}\mu = \binom{x\mu}{y\mu}.$$

*Proof* For every $z \in X_n$ we have

$$z\mu^{-1}\binom{x}{y}\mu = \begin{cases} y\mu, & z\mu^{-1} = x \\ z, & z\mu^{-1} \neq x \end{cases} = \begin{cases} y\mu, & z = x\mu \\ z, & z \neq x\mu \end{cases} = z\binom{x\mu}{y\mu},$$

and so $\mu^{-1}\binom{x}{y}\mu = \binom{x\mu}{y\mu}$. □

We now can state and prove the main result of the paper as follows:

**Theorem 4.7** *Let $S_n$ the symmetric group of $X_n$ and $U_\rho = \{\mu \in S_n : (x\rho)\mu = x\rho, x \in X_n\}$. Then $U_\rho$ is the unit group of $T(\rho, \preceq)$ and, for any $\varphi \in \operatorname{Aut} T(\rho, \preceq)$, there exists $\mu \in U_\rho$ such that $(\alpha)\varphi = \mu^{-1}\alpha\mu$ for all $\alpha \in T(\rho, \preceq)$.*

*Conversely, let $\mu \in U_\rho$. Then $\mu^{-1}\alpha\mu \in T(\rho, \preceq)$ for any $\alpha \in T(\rho, \preceq)$, and the map $\varphi : T(\rho, \preceq) \to T(\rho, \preceq)$ defined by $(\alpha)\varphi = \mu^{-1}\alpha\mu, \alpha \in T(\rho, \preceq)$ is an automorphism.*

*Proof* Let $\varphi \in \operatorname{Aut} T(\rho, \preceq)$. Then, first by Lemmas 4.5 and 4.6, there exists $\mu \in U_\rho$ such that

$$\left(\binom{x}{y}\right)\varphi = \binom{x\mu}{y\mu} = \mu^{-1}\binom{x}{y}\mu, \quad \text{for any } \binom{x}{y} \in E_\rho^w.$$

Secondly, we will prove that $(\alpha)\varphi = \mu^{-1}\alpha\mu$ for all $\alpha \in U_\rho$. For $\alpha \in U_\rho$, we must have $(\alpha)\varphi \in U_\rho$, and by Lemma 4.6, for any $\binom{x}{y} \in E_\rho^w$, we have

$$\begin{aligned} \binom{x\alpha\mu}{y\alpha\mu} &= \left(\binom{x\alpha}{y\alpha}\right)\varphi = \left(\alpha^{-1}\binom{x}{y}\alpha\right)\varphi \\ &= (\alpha^{-1})\varphi\left(\binom{x}{y}\right)\varphi(\alpha)\varphi = (\alpha)\varphi^{-1}\binom{x\mu}{y\mu}(\alpha)\varphi = \binom{x\mu(\alpha)\varphi}{y\mu(\alpha)\varphi}. \end{aligned}$$

Hence, $x\alpha\mu = x\mu(\alpha)\varphi$ and $y\alpha\mu = y\mu(\alpha)\varphi$. It follows that $(\alpha)\varphi = \mu^{-1}\alpha\mu$.

Finally, for any $\alpha \in T(\rho, \preceq) - E_\rho^w - U_\rho$, we have that $\alpha$ is a product of elements in $E_\rho^w \cup U_\rho$ by Proposition 2.5. It follows that $(\alpha)\varphi = \mu^{-1}\alpha\mu$ as required.

The converse part is obvious. □

*Remark* The $\mu$ is unique in Theorem 4.7. Indeed, let $\mu_1, \mu_2$ be elements of $U_\rho$ such that $\mu_1^{-1}\alpha\mu_1 = \mu_2^{-1}\alpha\mu_2$ for all $\alpha \in T(\rho, \preceq)$. Now, given any $x \in X_n$, we can choose an element $y \in X_n$ such that $\binom{x}{y} \in E_\rho^w$ or $\binom{y}{x} \in E_\rho^w$. For the former (Similarly for the latter), we have

$$\binom{x\mu_1}{y\mu_1} = \mu_1^{-1}\binom{x}{y}\mu_1 = \mu_2^{-1}\binom{x}{y}\mu_2 = \binom{x\mu_2}{y\mu_2}.$$

Hence $x\mu_1 = x\mu_2$, and so $\mu_1 = \mu_2$.

The map $\Psi : \operatorname{Aut} T(\rho, \preceq) \to U_\rho$ defined by

$$(\varphi)\Psi = \mu_\varphi, \quad \varphi \in \operatorname{Aut} T(\rho, \preceq), \quad \mu_\varphi \in U_\rho \quad \text{with } (\alpha)\varphi = \mu_\varphi^{-1}\alpha\mu_\varphi, \quad \forall \alpha \in T(\rho, \preceq)$$

is an isomorphism. We have

**Corollary 4.8** $\operatorname{Aut} T(\rho, \preceq) \cong U_\rho$.

Let $PT_n$ be the semigroup of partial transformations on $X_n$. Let $S_n^- = \{\alpha \in T_n : x\alpha \leq x, \forall x \in X_n\}$ [9]. Then, by Corollary 4.8, we have

**Corollary 4.9** $\operatorname{Aut} T_n \cong S_n$; $\operatorname{Aut} PT_n \cong S_n$ *and* $\operatorname{Aut} S_n^- \cong \{1_{X_n}\}$.



## References

1. Araújo, J., Konieczny, J.: Automorphism groups of centralizers of idempotents. J. Algebra **269**, 227–239 (2003)
2. Fountain, J.B.: Abundant semigroups. Proc. Lond. Math. Soc. **44**(33), 103–129 (1982)
3. Ganyushkin, O., Mazorchuk, V.: Classical Finite Transformation Semigroups: an Introduction. Algebra and Application, vol. 9 (2009)
4. Higgins, P.: Techniques of Semigroup Theory. Oxford University Press, Oxford (1992)
5. Howie, J.M.: Fundamentals of Semigroup Theory. Oxford University Press, Oxford (1995)
6. Schreier, J.: Über Abbildungen einer Abstrakten Menge auf ihre Teilmengen. Fundam. Math. **28**, 261–264 (1936)
7. Sullivan, R.P.: Automorphisms of transformation semigroups. J. Aust. Math. Soc. **20**, 77–84 (1975)
8. Šutov, È.G.: Homomorphisms of the semigroup of all partial transformations. Izv. Vysš. Učebn. Zaved., Mat. **3**, 177–184 (1961)
9. Umar, A.: On the semigroups of order-decreasing finite full transformations. Proc. R. Soc. Edinb., Sect. A, Math. **120**, 129–142 (1992)
10. Xiuliang, Y., Haobo, Y.: Maximal half-transitive submonoids of full transformation semigroups. Adv. Math. **40**(5), 580–586 (2011)